ON THE STRUCTURE OF A CURVILINEAR SHOCK WAVE

(O STRUKTURE KRIVOLINEINOI UDARNOI VOLNY)

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A.I.TOLSTYKH
(Moscow)

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The structure of shock waves for small Reynolds numbers is examined on the basis of Navier-Stokes equations. Appropriate functions are sought in the form of an expansion with respect to a small parameter which is proportional to curvature; some results are presented for zero-th and first terms of this expansion.

In connection with examination of supersonic flow around bodies at moderate Reynolds numbers the necessity arises to improve the accuracy of the pattern of inviscid flow according to which, in particular, the shock wave is considered as the surface of discontinuity. This question was first raised in the work [1], subsequently various authors referred to this problem in simplified form [2 to 4]; a successive study of this question is contained in [5]. The method developed in [5] consists of constructing "internal" asymptotic expansions with respect to a small parameter which tends to zero together with viscosity. The terms of the asymptotic expansions correspond to discontinuous "external" flow and to the "internal" structure of the shock. However, a different approach which is examined in the present note is possible (*).

Let us assume that the flow in the entire infinite region is described by Navier-Stokes equations in which a characteristic measure of viscosity, for instance the viscosity at infinity, plays the part of a fixed parameter. The problem consists of determination of the structure of a curvilinear shock wave under the condition that behind it viscous tensions and thermal fluxes exist. In other words, it is required to find a solution of Navier-Stokes equations such that at infinity it tends to approach the values of parameters in undisturbed flow and such it satisfies the required number of boundary conditions on a sufficiently smooth Γ curve (for instance, values of velocity derivatives and temperature derivatives on Γ may be given for the solution).

In the one-dimensional case (rectilinear shock) it is possible to integrate the system of Navier-Stokes equations directly and to solve the given problem (**). In connection with this it is natural to look for a solution of the general problem in the form of an expansion with respect to a small

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*) It can be demonstrated that the results obtained below may be utilized in the solution of the problem of viscous fluid flow over a body.

**) For this one-dimensional system the specific solution, which at infinity on both sides of the shock approaches definite values, is known as the solution of Becker [6].

parameter which is proportional to the curvature (or inversely proportional to the radius of curvature).(*) The zero-th order term of this expansion here will turn out to be the structure of a rectilinear shock behind which velocity and temperature gradients exist. It is important to note that in connection with the absence of singularity ("edge effect") when this parameter tends to zero, the necessity disappears for the construction of two asymptotic expansions, an "internal" and "external", which are indispensible in the case when viscosity is allowed to approach zero [5].

For simplicity limiting ourselves to the two-dimensional case, we introduce curvilinear orthogonal coordinates (n and s) connected with the curve Γ (s is taken along Γ, n is directed towards the concave side of the curvature of Γ), which characterizes the "form" of the shock in the sense that the normal n, in distinction from the tangent s, will be the direction of rapid change of the appropriate functions. Denoting by R_0 the characteristic value of the radius of curvature of Γ, we introduce a characteristic measure of length δ in the direction of the normal. The dimensionless parameter $\varepsilon = \delta/R_0$ is generated. Assuming that $n^\circ = n/\delta$, $s^\circ = s/R_0$, and having written the complete system of Navier-Stokes equations in the coordinates (n°; s°), we will be looking for a solution in the form

$$f = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \qquad (1)$$

where f is any of the desired functions.

In the zero-th approximation a system of the following form is obtained

$$\frac{d}{dn}(\rho^{(0)} v^{(0)}) = 0 \qquad (2)$$

$$\rho^{(0)} v^{(0)} \frac{du^{(0)}}{dn} = \frac{d}{dn}\left(\mu^{(0)} \frac{du^{(0)}}{dn}\right), \qquad \rho^{(0)} v^{(0)} \frac{d\theta^{(0)}}{dn} = -\frac{dp^{(0)}}{dn} + \frac{4}{3}\frac{d}{dn}\left(\mu^{(0)} \frac{dv^{(0)}}{dn}\right)$$

$$\rho^{(0)} v^{(0)} \frac{d\theta^{(0)}}{dn} = \frac{1}{\sigma}\frac{d}{dn}\left(\mu^{(0)} \frac{d\theta^{(0)}}{dn}\right) + \frac{d}{dn}\left(\mu^{(0)} u^{(0)} \frac{du^{(0)}}{dn}\right) + \left(\frac{4}{3} - \frac{1}{\sigma}\right)\frac{d}{dn}\left(\mu^{(0)} v^{(0)} \frac{dv^{(0)}}{dn}\right)$$

$$\theta^{(0)} = h^{(0)} + \frac{1}{2}(v^{(0)})^2, \qquad p^{(0)} = \frac{\varkappa - 1}{\varkappa}\rho^{(0)} h^{(0)} \qquad (-\infty \leqslant n \leqslant 0)$$

Here u, v, p, ρ, h and μ are velocity components along s and n, pressure, density, enthalpy and coefficient of viscosity respectively; σ and $\varkappa$ are the Prandtl number and adiabatic exponent The boundary conditions for these equations have the form (**)

$$\begin{gathered} u^{(0)} \to u_\infty, \quad v^{(0)} \to v_\infty, \quad p^{(0)} \to p_\infty, \quad h^{(0)} \to h_\infty \quad \text{for} \quad n \to -\infty \\ \frac{du^{(0)}}{dn} = \varphi_1(s), \quad \frac{dv^{(0)}}{dn} = \varphi_2(s), \quad \frac{dh^{(0)}}{dn} = \varphi_3(s) \quad \text{for} \quad n = 0 \end{gathered} \qquad (3)$$

From results of numerical integrations of systems (2) and (3) the magnitude of parameter ε can be established. The solution of this system which is determined by one dimensionless variable

$$\xi = \rho_\infty v_\infty \int_0^n \frac{dn}{\mu}$$

gives an effective thickness of the shock wave $\Delta\xi$, at $n = 0$ independent of boundary conditions. As an illustration of this, typical profiles of the

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*) Here and in what follows it is assumed that the radius of curvature appears as the characteristic scale of length for the variation of function along the shock.

**) In the following the solution will be sought with accuracy to derivatives of functions without sacrifice in generality.

normal velocity component $V = v^{(0)}/\sqrt{h_\infty}$ are shown in Fig. 1, in dimensionless form. These profiles were obtained at Prandtl·number $\sigma = \frac{3}{4}$ for various values of derivatives $d\mu^{(0)}/d\xi$ and $dh^{(0)}/d\xi$ at the point where $dv^{(0)}/d\xi = 0$.(*) Transferring to the physical veriable n, it can be shown that

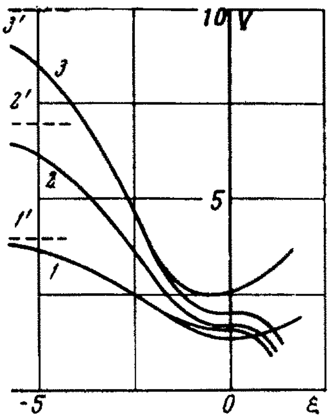

Fig. 1

$$\frac{\delta}{R_0} \sim \frac{\mu_0}{\rho_\infty v_\infty R_0} = \frac{1}{N_{\mathrm{Re}}}$$

where μ_0 is the viscosity at stagnation temperature. Taking some liberty with the selection of characteristic scale factors we set $\varepsilon = 1/N_{Re}$. Now Expression (1) will have the form of an asymptotic expansion with respect to parameter $1/N_{Re}$ which tends to approach zero. In fact, however, (1) is different in principle from the corresponding expansion for disappearing viscosity.

For subsequent approximations, systems of ordinary linear equations are obtained. The constants of integration of these equations are completely determined from boundary values.

Corrections to the classical shock structure (i.e. Becker's solution) and consequently to the common shock wave relationships arise from terms in expansion (1), (a) due to the presence of gradients of desired functions (i.e. due to the presence of viscous tensions and thermal currents) immediately behind the shock wave, and (b) as a result of distortion of the shock wave; corrections of the second kind appear only in the first and subsequent approximations $f^{(k)}$ $(k \geqslant 1)$.

In the general case numerical calculations are required for determination of the function $f^{(k)}$; however, if the corresponding gradients do not have orders higher than f_∞/R_0 (f_∞ is the value of f at infinity) then it is possible to obtain coefficients with ε in the form of quadratures by means of Becker's solution.

Final results are considerably simplified if one takes advantage of the hypersonic approximation pointed out in [7], and assumes the temperature at infinity to be equal to zero (in the sum $f^{(0)} + \varepsilon f^{(1)}$ here terms which are of the order $\varepsilon M_\infty{}^2$, where M_∞ is the Mach number of the unperturbed flow, are not taken into account).

The angle between the velocity vector at infinity and the direction of the normal is designated by ϑ, $R(\vartheta)$ is the radius of curvature of the natural shock front [7].

Let us assume that

$$R_0 = R\,(0),\ N_{\mathrm{Re}} = \mu_0 / \rho_\infty v_\infty\,(0)\,R_0$$

Then for values of $\sigma = \frac{3}{4}$ and $\mu \sim h$, we obtain for the first terms of the expansion of nondimensional velocity components U and V, of enthalpy H and of density P

$$U = \frac{u}{w_\infty \sin\vartheta}, \quad V = \frac{v}{w_\infty \cos\vartheta}, \quad H = \frac{h}{w_\infty{}^2}, \quad \mathrm{P} = \frac{\rho}{\rho_\infty} \qquad (w_\infty{}^2 = u_\infty{}^2 + v_\infty{}^2)$$

the following relationships:

*) With respect to the order of magnitude these derivatives correspond to conditions of hypersonic flow over blunt bodies at small Reynolds numbers. Asymptotes 1', 2' and 3' belonging to families of profiles 1, 2 and 3 correspond to values of $V_\infty = 3$; 7 and 10.

$$\frac{1}{2}\cos\vartheta\,(1-v_0^2)\frac{dU^{(1)}}{dn^\circ}-U^{(1)}=$$
$$=\frac{R_0}{R}\cos\vartheta\left\{\nu M(v_0)-\frac{1}{2}\left(\frac{1}{3}v_0^3-v_0^2+v_0-\frac{1}{3}\right)-\frac{1}{3}\,\frac{[F(v_0)-\nu_1](v_0-\gamma)(1-v_0)}{v_0}\right\}$$

$$\frac{2}{3}\cos\vartheta\,(1-v_0^2)\frac{dV^{(1)}}{dn^\circ}+\frac{1}{1+\gamma}\,\frac{v_0^3-(1+2\gamma)v_0^2-\gamma v_0+1}{v_0^2(1+v_0)}V^{(1)}=$$
$$=\frac{R_0}{R}\left\{\frac{2}{1+v_0}\frac{\theta^{(1)}}{\cos^2\vartheta}+\nu\frac{v_0^2+\gamma}{v_0^2}N(v_0,\vartheta)+\frac{2}{3}\cos\vartheta\,(v_0^2-1)+\right.$$
$$\left.+\nu\frac{\sin^2\vartheta}{\cos\vartheta}\left[\frac{v_0^2}{2}-\gamma v_0+(\gamma^2-1)\ln(v_0-\gamma)-\nu_2\right]\right\}$$

$$\frac{2}{3}\cos\vartheta\,(1-v_0^2)\frac{d\theta^{(1)}}{dn^\circ}-\theta^{(1)}=\frac{R_0}{R}\cos^2\vartheta\left\{\sin\vartheta\tan\vartheta\left[\nu M(v_0)+\frac{1}{2}(v_0^2-1)\right]+\right.$$
$$\left.+\frac{1}{3}\cos\vartheta\left(v_0^4+\frac{4}{3}v_0^3-v_0^2-2v_0+\frac{2}{3}\right)\right\}$$

$$\mathrm{P}^{(1)}=\frac{R_0}{R}\nu\frac{N(v_0,\vartheta)}{v_0}-\frac{V^{(1)}}{v_0^2}$$

Here

$$\theta^{(1)}=H^{(1)}+v_0V^{(1)}$$

$$F(t)={}^1/_2\,t^2+(\gamma+1)\,t+\gamma(\gamma+1)\ln(t-\gamma)$$

$$M(t)={}^1/_2\,\gamma(1-\gamma)^{-1}(1-t^3)+{}^1/_2\,\gamma(1-t^2)+\nu_3(1-t)-$$
$$-(1-\gamma^2)(1-3\gamma)\ln(t-\gamma)+t^{-1}\gamma^2(t^2-1)\ln(t-\gamma)+t^{-1}\nu_1-\nu_4$$

$$N(t,\vartheta)=\cos\vartheta\,[{}^1/_2t^2-\gamma t+(\gamma^2-1)\ln(t-\gamma)-\nu_2]+$$
$$+\sin\vartheta\,[{}^1/_2t+(\gamma+1)t^{-1}(t-\gamma)\ln(t-\gamma)+t^{-1}\nu_1-\nu_5]$$

$$\gamma=(\varkappa-1)/(\varkappa+1),\qquad \nu={}^2/_3(1+\gamma),\quad \nu_1={}^3/_2+\gamma+\gamma(\gamma+1)\ln(1-\gamma)$$

$$\nu_2={}^1/_2-\gamma+(\gamma^2-1)\ln(1-\gamma)$$

$$\nu_3={}^3/_2+2\gamma+\gamma(\gamma+1)^{-1}(6\gamma^2+6\gamma-5)+\gamma(\gamma+1)\ln(1-\gamma)$$

$$\nu_4=\nu_1-(1-\gamma^2)(1-3\gamma)\ln(1-\gamma),\quad \nu_5=\nu_1+{}^1/_2+(1-\gamma^2)\ln(1-\gamma)$$

v_0 is a function which is determined by the relationship

$$\nu_1-F(v_0)=n^\circ$$

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